

A Birkhoff theorem for Shape Dynamics

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Abstract

Shape Dynamics is a theory of gravity that replaces refoliation invariance for spatial Weyl invariance. Solutions of the Einstein equations that have global constant mean curvature slicings have equivalent solutions in Shape Dynamics, but there are solutions of Shape Dynamics that have no counterpart in GR, just as there are solutions of GR that have no global constant mean curvature slicings. It is interesting then to analyze directly the equations of motion of Shape Dynamics to find its own solutions, irrespective of properties of known solutions of GR. Here we use these equations in a spherically symmetric, asymptotically flat ansatz to derive an analogue of the Birkhoff theorem. There are three significant differences with respect to the usual Birkhoff theorem in GR. The first regards the posing of the problem: in Shape Dynamics we must establish from the start the boundary conditions of our phase space variables. The second difference regards the construction of the solution: we heavily use the Shape Dynamics spatial Weyl gauge freedom to simplify the problem. The remaining, and most important difference is that the solution obtained is uniquely the isotropic wormhole solution, in which no singularity is present, as opposed to maximally extended Schwarzschild. This provides an explicit example of the breaking of the duality between General relativity and Shape Dynamics, and exhibits some of its consequences.

1 Introduction

1.1 Shape Dynamics

Shape Dynamics is a theory of gravity, formulated in the Hamiltonian 3+1 formalism. It possesses two dynamical propagating degrees of freedom, and has as kinematical variables the same g_{ab} and π^{ab} as the Hamiltonian version of ADM General Relativity. What is noteworthy about it is that it possesses spatial Weyl invariance, acting on both the metric and on the momenta. It maintains the correct number of degrees of freedom since it does not have refoliation invariance.¹

Shape Dynamics is intimately related to specific gauge-fixings of Hamiltonian 3+1 ADM. It is a theory that takes as its geometric observables *spatial* conformal-diffeomorphism invariants, as opposed to *space-time* diffeomorphism invariants. The gauge-fixings of ADM that are related to Shape Dynamics are either constant mean curvature (CMC) for the closed spatial manifold case, or maximal slicing for the open manifold case. To be more precise, suppose that we restrict our spatial manifolds Σ - the spatial part of globally hyperbolic spacetimes $\Sigma \times \mathbb{R}$ - to be open (not necessarily compact). Then Shape Dynamics is a gauge theory whose reduced phase space is intimately connected to a maximal slicing $g_{ab}\pi^{ab} = \pi = 0$ gauge fixing of ADM. There is a special property of these gauge-fixings which is what relates them to Shape Dynamics. That property is that they also moonlight as generators of spatial Weyl transformations. In the same way that $\pi^{ij}_{;j} = 0$ generates spatial diffeomorphisms in Hamiltonian ADM general relativity, $\pi = 0$ (maximal slicing) generates spatial Weyl transformations, while $\pi - \langle \pi \rangle \sqrt{g} = 0$ (CMC) generates total-volume preserving conformal transformations.

This is not to say that we are looking simply for maximal slicings of given space-time solutions of GR, since there are possible obstructions to space-time solutions (such as degeneracy of the 4-metric) that cannot be seen (i.e. present no analogous obstruction) in the phase space solution $(g_{ab}(t), \pi^{ab}(t))$ of the equations of motion of Shape Dynamics. On the other hand, if a given space-time has a complete maximal slicing it will be also a solution of the gauge-fixed reduced theory, whereas if the complete slicing is not attainable, the space-time will have no counter-part in the reduced theory. The point of the matter is that any solution of the gauge-fixed maximal-slicing reduced theory has a counterpart in Shape Dynamics, which means it has a Weyl invariant representation. Thus from the point of view of Shape Dynamics, a global space-time satisfying Einstein's equation is physical only if it possesses a global maximal slicing.

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¹To be more precise, it does not possess the corresponding phase space symmetry, since in phase space refoliations are not really meaningful.

The breaking of the duality between the two theories is represented by a collapse of the required space-time distance between the hypersurfaces needed to propagate the maximal slicing condition. However, *this collapse need not be physical in either of the two theories*. In both sides, it is a gauge-dependent statement. Nonetheless, the *translation* between the two theories, which have different gauge symmetries to explore, *does* break down when the lapse collapses, and the two theories can have different continuations from then on. As we will see, there are solutions of Shape Dynamics which are only locally isomorphic to space-time solutions of General Relativity. They do not form space-times, but are by definition still physical in the Shape Dynamics sense.

We will find such a solution, and furthermore show that it is the unique such solution if we assume asymptotic flatness and spherical symmetry in vacuum. This imposition of boundary conditions already signals a difference between the Shape Dynamics analogue of the Birkhoff theorem and the GR one, which yields a static 4-metric from spherical symmetry irrespective of the boundary conditions. Further differences lie at the construction level, since we use the Shape Dynamics equations of motion and its respective Weyl freedom. Finally, the way in which the actual spherically symmetric solution for Shape Dynamics differs is that it represents an isotropic wormhole solution, as opposed to (maximally extended) Schwarzschild.

1.2 Birkhoff's theorem in GR

In General Relativity, Birkhoff's theorem [1]² states that a spherically symmetric solution of the Einstein field equations in vacuum $R_{\mu\nu} = 0$, must contain an extra Killing vector field. The resulting solution is given by the Schwarzschild metric:

$$ds^2 = -\frac{1}{1 - \frac{2m}{r}}dt^2 + \left(1 - \frac{2m}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \quad (1)$$

If the Killing vector field is time-like, the resulting metric will be locally static and asymptotically flat, corresponding to the region $r > 2m$ in (1). If it is space-like, it will be homogeneous and it will run into a singularity at $r = 0$, corresponding to the region $r < 2m$ in (1).

Clearly something goes wrong at $r = 2m$ in (1): the radial component g_{rr} of the metric blows up, while g_{tt} vanishes. However, this is a *coordinate singularity*, since no physical observable, such as invariants of the curvature, blows up in that region.

Coordinate invariance and the degeneracy of the 4-metric

To characterize some object as coordinate invariant we must first define what we take to be valid coordinate transformations. As usual, we define it as a smooth diffeomorphism: a smooth transformation with a smooth inverse. The requirement of existence of the inverse implies that to be *characterized as a coordinate transformation the transformation's Jacobian has to have a finite, non-zero determinant*. A coordinate change which obeys these conditions is used to obtain a maximal extension of Schwarzschild (Kruskal-Szekeres) which avoids the pathologies of the original coordinate system employed in (1). For future purposes we call attention to the fact that the *determinant* of the 4-metric of the Schwarzschild solution above (1) $\det g_{\mu\nu}$ radially vanishes only at the singularity $r = 0$ (although the g_{00} component vanishes at the horizon). Unlike what occurs in the above case however, a 4-metric which is degenerate in some region (i.e. whose determinant vanishes in that region) cannot be transformed into a regular (non-degenerate) 4-metric by a viable coordinate transformation. This distinction will become important in order to show that the Shape Dynamics solution *is not* a space-time solution.

2 Shape Dynamics for asymptotically flat conditions

2.1 Construction

The construction of Shape Dynamics for the case of a closed spatial manifold is more complicated than the one we are about to present here [3]. That is because in the open case we have boundary conditions that allow us to define a non-trivial Hamiltonian for Shape Dynamics, whereas in the closed case we are forced to use more subtle means. Since the construction of the theory for the case where the underlying spatial manifold has boundary is much simpler, we will present it here without reference to the changes that have to be made in the closed manifold case.

The first step is to write out the constraints of canonical GR in its 3+1 ADM form:

$$S(x) := \frac{G_{abcd}\pi^{ab}\pi^{cd}}{\sqrt{g}}(x) - R(x)\sqrt{g}(x) = 0 \quad (2)$$

$$H_a := \pi_b^a{}_{;a} = 0 \quad (3)$$

²In fact, the theorem today known as Birkhoff's theorem was first discovered by Jebsen in [2].

where the points x belong to an open 3-manifold Σ , g_{ab} is the spatial 3-metric and its conjugate momenta π^{ab} (intimately related to the extrinsic curvature of a foliation). The scalar constraint (2) generates on-shell refoliations of spacetime, while the momentum constraint generates foliation preserving diffeomorphisms. The second step is to extend phase space in a trivial manner, including the variables ϕ and its canonically conjugate momenta π_ϕ . This entails the appearance of third set of first class constraints

$$\pi_\phi = 0$$

Now we perform a canonical transformation in the extended phase space with coordinates $(g_{ab}, \pi^{ab}, \phi, \pi_\phi)$ of the form:

$$t_\phi : (g_{ab}, \pi^{ab}, \phi, \pi_\phi) \mapsto (e^{4\psi} g_{ab}, e^{-4\psi} \pi^{ab}, \phi, \pi_\phi - 4\pi)$$

where $\pi = g_{ab}\pi^{ab}$. The extra first class constraint in this extended theory after the canonical transformation is given by $\pi_\phi - 4\pi \approx 0$. This constraint is associated to the trivial symmetry generated by simultaneously transforming the metric canonical variables and a compensating transformation for the conformal factor: $(g_{ab}, \pi^{ab}) \mapsto (e^{4\psi} g_{ab}, e^{-4\psi} \pi^{ab})$ accompanied by the transformation $(\phi, \pi_\phi) \mapsto (\phi - \psi, \pi_\phi)$ keeps the combination $(e^{4\phi} g_{ab}, e^{-4\phi} \pi^{ab})$ invariant (it is this combination that enters the action in extended phase space).

The smeared scalar constraint (2) becomes, for $\phi = \ln \Omega$

$$t_\psi S(N) = \int_\Sigma \left(\nabla^2 \Omega + R\Omega - \frac{1}{8} \pi^{ab} \pi_{ab} \Omega^{-7} \right) \approx 0 \quad (4)$$

Ignoring boundary terms (see [5] for details on how to treat the boundary terms), the smeared diffeomorphism constraint becomes

$$t_\phi H_a(\xi^a) = \int_\Sigma (\pi^{ab} \mathcal{L}_\xi g_{ab} + \pi_\phi \mathcal{L}_\xi \phi) d^3x \approx 0 \quad (5)$$

while we now have a new constraint which was not present in ADM:

$$C(\rho) = \int_\Sigma (\pi_\phi - 4\pi) \rho \approx 0 \quad (6)$$

Where N , ξ^a and ρ are the Lagrange multipliers multiplying respectively the extended scalar constraint $t_\phi S(x)$, the extended momentum constraint $t_\phi H_a(x)$ and the new ‘‘conformal’’ constraint $C(x)$.

As is usual for dynamical systems on manifolds that have a boundary, we must add terms to the total Hamiltonian to ensure differentiability and thus the well-posedness of the equations of motion. These boundary terms will depend on the boundary conditions imposed on the phase space variables, and the boundary conditions themselves have to be preserved by the equations of motion. The complete treatment is given in [5].

Let us summarize here some of the assumptions and results of the boundary treatment. First, the boundary conditions taken for the asymptotically flat metric variables and Lagrange multipliers of the scalar and diffeomorphism constraint are:

$$\begin{aligned} g_{ab} &\rightarrow \delta_{ab} + \mathcal{O}(r^{-1}) & , & & \pi^{ab} &\rightarrow \mathcal{O}(r^{-2}) \\ N &\rightarrow 1 + \mathcal{O}(r^{-1}) & , & & \xi^a &\rightarrow \mathcal{O}(1) \end{aligned} \quad (7)$$

The fall-off condition for the remaining Lagrange multiplier ρ can be determined from the requirements of consistency and finiteness of the boundary terms. In [5] we have shown that in order to preserve the asymptotic form of the metric variables and obtain finite boundary terms, the asymptotic behavior of the Lagrange multiplier and the conformal variables should be $\rho \rightarrow \mathcal{O}(r^{-1})$, and $e^\phi \rightarrow 1 + \mathcal{O}(r^{-1})$ and $\pi_\phi \rightarrow \mathcal{O}(r^{-2})$.

To recover the ADM dynamical system, one must merely gauge-fix this extended theory by setting $\phi = 0$. For the construction of Shape Dynamics, we perform the gauge-fixing $\pi_\phi = 0$ on the extended system. The only constraint that does not commute with this gauge-fixing is exactly (4). This constraint can be solved for Ω [6] in terms of the variables g_{ab} and π^{ab} , and thus together with the setting of $\pi_\phi = 0$ the system is reduced, and the Dirac bracket also reduces to the canonical Poisson brackets of the variables (g_{ab}, π^{ab}) . It turns out that in our present case there remains a total Hamiltonian residing on the boundary $\partial\Sigma$. It is of the form:

$$H_{\text{SD}}[g, \pi] = - \int_{\partial\Sigma} d^2y \sqrt{h} (2(k - k_o) + 8r_e \Omega_o^e) \quad (8)$$

where k is the trace of the extrinsic curvature of the boundary, h_{ab} is the metric at the boundary and r^a is the

normal to the boundary. The other remaining constraints are the usual spatial diffeomorphism constraints³

$$\int d^3x (\pi^{ab} \mathcal{L}_\xi g_{ab}) = 0$$

with ξ^a now respecting the asymptotically flat boundary conditions. Finally, from the reduction of the constraint $C = \pi_\phi - 4\pi$ we also obtain a Weyl (or conformal) constraint $\pi = 0$ (whose smearing ρ also must satisfy the appropriate boundary conditions).

2.2 Equations of motion

Shape Dynamics is obtained from a Linking theory (what we have called so far the extended theory), where all quantities are local. Thus the simplest way to formulate Shape Dynamics' equations of motion, boundary charges, counter-terms and fall-off conditions is to consider these in the larger setting of the Linking theory, and then use phase space reduction.

For the open case, where we institute maximal slicing as opposed to constant mean curvature slicing (CMC) which we institute when the spatial manifold is closed, the canonical transformation of the metric variables are given by $(g_{ab}, \pi^{ab}) \mapsto (e^{4\phi} g_{ab}, e^{-4\phi} \pi^{ab})$.

As in the CMC case, although the lapse does not figure in the fundamental equations of Shape Dynamics, we can use an identity obtained by phase space reduction, which requires an effective lapse N_o , the solution of

$$e^{-4\phi_o} (\nabla^2 N_o + 2g^{ab} \phi_{,a}^o N_{,b}^o) - N_o e^{-6\phi_o} G_{abcd} \pi^{ab} \pi^{cd} = 0 \quad (9)$$

where we have denoted the solution of (4) by $\Omega_o[g, \pi] = e^{\phi_o[g, \pi]}$.

The equations valid in the present case are:

$$\dot{g}_{ab} = 4\rho g_{ab} + 2e^{-6\phi_o} \frac{N_o}{\sqrt{g}} \pi^{ab} + \mathcal{L}_\xi g_{ab} \quad (10)$$

$$\begin{aligned} \dot{\pi}^{ab} = & N_o e^{2\phi_o} \sqrt{g} \left(R^{ab} - 2\phi_o^{;ab} + 4\phi_o^{;a} \phi_o^{;b} - \frac{1}{2} R g^{ab} + 2\nabla^2 \phi_o g^{ab} \right) \\ & - \frac{N_o}{\sqrt{g}} e^{-6\phi_o} \left(2(\pi^{ac} \pi_c^b) - \frac{1}{2} (\pi^{cd} \pi_{cd}) g^{ab} \right) \\ & - e^{2\phi_o} \sqrt{g} \left(N_o^{;ab} - 4\phi_o^{;a} N_o^{;b} - \nabla^2 N_o g^{ab} \right) + \mathcal{L}_\xi \pi^{ab} - 4\rho \pi^{ab} \end{aligned} \quad (11)$$

Note the presence of the conformal gauge terms $4\rho g_{ab}$ and $-4\rho \pi^{ab}$.

We also pause to mention that equations of motion for Shape Dynamics in the CMC case are considerably more complicated by the fact that the conformal factor $\hat{\phi}$ must be total volume preserving, and thus depends on g , entering the variations. We will refrain from writing these equations of motion down here, since we are not concerned with the CMC case.

The Weyl invariance of the equations of motion are manifest by the fact that the combinations $e^{4\phi_o[g, \pi]} g_{ab}$, $e^{-4\phi_o[g, \pi]} \pi^{ab}$ are conformally invariant, and it is these combinations that enter the equations. To obtain a solution of ADM, we must impose the gauge-fixing $\phi_o[g, \pi] = 0$, which is a condition of course on g_{ab} and π^{ab} .

3 Birkhoff's theorem in Shape Dynamics

One of the key *advantages* of Shape Dynamics is that we can use the conformal gauge to absorb terms that we would otherwise be unable to. Starting with (10) we will use this freedom to cancel all terms proportional to the metric. To obtain the non-trivial Minkowski solution, we must include a non-zero mass in the assumptions. The boundary conditions are such that the energy of the solution - defined in (8) - is given by m , which imposes a particular form for the next to leading order asymptotics of the conformal factor Ω . The mass of the solution is included at the origin as a $\delta(r)$ term. However, a matter term on rhs of equation (4) appears as $16\pi\Omega^5 p$ where p is the energy density of the source we are implementing (and here π is just the irrational number, not the trace of the gravitational momenta). This implies that there is proportionality factor between p and m different than what would normally happen in a Schwarzschild solution.

³In Shape Dynamics the constraints decouple the conformal factor from the diffeomorphism constraint. This occurs because the diffeomorphism generator decouples into one part that is the original diffeomorphism generator and another that gets absorbed by the conformal constraint upon reduction:

$$\int d^3x \pi^{ab} e^{-4\phi} \mathcal{L}_\xi (e^{4\phi}) g_{ab} = \int d^3x (\pi^{ab} \mathcal{L}_\xi g_{ab} + 4\pi \mathcal{L}_\xi \phi)$$

. The point is that we can absorb the term $4\mathcal{L}_\xi \phi$ into the Lagrange multiplier ρ for the remaining conformal constraint (i.e. it vanishes on-shell).

Construction

We start by defining an objective criterion to identifying conformally flat metrics, namely the Cotton tensor. Much like the Weyl tensor in 4-dimensions, the Cotton tensor encodes the conformally invariant degrees of freedom of the 3-metric g_{ij} . As is easy to verify, spherical symmetry implies that the Cotton tensor for the 3-metric vanishes, thus there is no coupling to the Weyl-invariant degrees of freedom, except through the boundary conditions.

Thus our metric is conformally flat, and we restrict ourselves to a phase space in which $g_{ab} = \Omega^4(r, t)\eta_{ab}$ where $\eta_{ab} = \text{diag}(1, r^2, r^2 \sin^2(\theta))$ in the spherical coordinates $\{r, \theta, \varphi\}$. Let us reproduce (10) here for the reader's convenience:

$$\dot{g}_{ab} = 4\rho g_{ab} + 2e^{-6\phi_o} \frac{N_o}{\sqrt{g}} \pi^{ab} + \mathcal{L}_\xi g_{ab}$$

We can replace g_{ab} by η_{ab} , ϕ by $\ln \Omega$, and $\dot{g}_{ab} = 4\Omega^3 \dot{\Omega} \eta_{ab}$ on the lhs.

For generality, we don't want to assume the shift is zero. In accordance with spherical symmetry for simplicity we assume that it is of the form $\xi^a = \xi(r, t)\delta_r^a$. The Lie derivative of the metric with respect to a vector field of this form is:

$$\mathcal{L}_\xi(\tilde{\Omega}\eta_{ab}) = \xi \tilde{\Omega}_{,r} \eta_{ab} - 2r \tilde{\Omega} \xi (\delta_a^\theta \delta_b^\theta + \sin^2(\theta) \delta_\varphi^a \delta_\varphi^b) + 2\tilde{\Omega} \xi_{,r} \delta_a^r \delta_b^r \quad (12)$$

where for convenient notation we wrote $\tilde{\Omega} = \Omega^4$. We can rewrite this in a form which has more terms proportional to the metric:

$$2r \tilde{\Omega} \xi (\delta_a^\theta \delta_b^\theta + \sin^2(\theta) \delta_\varphi^a \delta_\varphi^b) = 2\eta_{ab} \frac{\tilde{\Omega} \xi}{r} - 2 \frac{\tilde{\Omega} \xi}{r} \delta_a^r \delta_b^r$$

Now putting all the terms proportional to the metric on the lhs of (10) we get:

$$(f(r, t) - \rho)\eta_{ab} = \gamma(r, t)\sigma_{ab} + 2\tilde{\Omega} \delta_a^r \delta_b^r (\xi_{,r} - \frac{\xi}{r}) \quad (13)$$

where $f(r, t) = 4\Omega^3 \dot{\Omega} + \frac{\tilde{\Omega} \xi}{r}$ which clearly decays faster than $1/r$, $\gamma(r, t)$ is an unimportant collection of the prefactors of the trace-free part of the momenta, which we have defined as $\sigma^{ab} := \pi^{ab} - \frac{1}{3}\pi g^{ab}$.

We have still some gauge freedom to explore, and we do so by choosing $\rho(r, 0) = f(r, 0)$. Then the lhs of (10) is identically zero, and taking the trace of the equation we obtain $\xi_{,r} - \frac{\xi}{r} = 0$, which means $\xi = \alpha r$ and, more importantly, that the momenta vanish: $\sigma_{ab}|_{t=0} = 0$.

As shown in [5], if we input a point source p at $r = 0$ in radial coordinates the scalar equation (4) becomes $\partial^2 \Omega = 2\pi p \delta(r) \Omega(0)^5$, where the 2π factor is exactly what appears from the scalar constraint (2π comes from $16\pi/8$, where the 8 is the factor for $\nabla^2 \Omega$ in (4)). The solution of this equation with the general boundary conditions $1 + \mathcal{O}(1/r)$ is given by $\Omega = 1 + \frac{m}{2r}$ where $m = p\Omega(0)^5$. As $\Omega(0)$ is a divergent quantity, some sort of regularization should take place.⁴ To see that the boundary conditions on the next to leading order contribution to Ω are justified, we calculate the asymptotic energy, given by (8). Trivially, one obtains $\partial_r \phi \rightarrow -\frac{m}{2r^2}$, and asymptotically $k \rightarrow \partial_j g_{ij} - \partial_i(\delta^{kl} g_{kl})$ which is equal to zero since $g_{ij} = \delta_{ij}$. Thus we are left with (after restoration of the $\frac{1}{16\pi}$ factor):

$$E_{\text{SD}} = -\frac{1}{2\pi} \int_{\partial\Sigma} d^2 y \partial_r \phi (r^2 \sin(\theta) d\theta d\phi) = m \quad (14)$$

Moving on, the general solution to (9) with asymptotically flat boundary conditions becomes $N = 1 - \frac{2b}{m+2r}$, where b is an integration constant. If we input the lapse $N = 1 - \frac{2b}{m+2r}$ and $\phi = \ln(1 + \frac{m}{2r})$ into (11) we obtain

$$\dot{\pi}^{ab} = (1 + \frac{m}{2r})^2 \begin{pmatrix} \frac{8(-1+b)m}{r(m+2r)^2} & 0 & 0 \\ 0 & -\frac{4(-1+b)m r \sin^2(\theta)}{(m+2r)^2} & 0 \\ 0 & 0 & -\frac{4(-1+b)m r}{(m+2r)^2} \end{pmatrix} \quad (15)$$

We see that to maintain the boundary conditions on the metric momenta, $\pi^{ab} \sim \mathcal{O}(r^{-2})$ we must have $b = 1$, thus $\dot{\pi}_{t=0}^{ab} = 0$, and in fact $\pi^{ab}(t) = 0$.⁵ Thus, we find the solution compatible with our boundary conditions by taking $b = 1$.

⁴ p could be formally defined by a limiting procedure, but in this case we would have a product of distributions in the defining equation (4), which might be ill-defined.

⁵We could have derived this directly had we used more of our gauge freedom to require $\rho(r, t) = f(r, t)$ above, instead of the weaker $\rho(r, 0) = f(r, 0)$.

To reconstruct the 4-metric from this, we must gauge-fix $\phi_o = 0$, or in other words $\Omega_o = 1$. We use the Weyl invariance of $e^{4\phi_o[g,\pi]}g_{ab}$, $e^{-4\phi_o[g,\pi]}\pi^{ab}$ and go to the gauge where $\Omega_o = 1$. Rewriting the lapse solution as $1 - \frac{2m}{m+2r} = \frac{1-\frac{m}{2r}}{1+\frac{m}{2r}}$, the reconstructed space-time from this is:

$$ds^2 = - \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}} \right)^2 dt^2 + \left(1 + \frac{m}{2r} \right)^4 (dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)) \quad (16)$$

In the no back-reaction limit, the dynamics of particles in Shape Dynamics will act as if obeying the geometry of this space-time, which as we saw, up to the gauge symmetry of Shape Dynamics, is the unique form of the translation of a solution for an asymptotically flat, spherically symmetric ansatz in Shape Dynamics into a space-time.

4 Conclusion

Here we explore a few of the consequences of the solution, mainly transcribed from [8], and point out some of the differences in the method utilized in the obtention of the solution here and those utilized for the obtention of the usual Birkhoff theorem. We leave a more careful analysis of the consequences for the upcoming paper [7], where we show also that this solution will form from collapse of a certain homogeneous star model.

New elements regarding the method

We start by pointing out that in the construction of the solution, unlike what is done in the usual Birkhoff theorem, we utilize both the boundary conditions and our ability to absorb the pure gauge terms in ρ . For instance we have used the assumption that $\mathcal{O}(\xi^a) < \mathcal{O}(1)$. We have also assumed the relevant fall-off rates for the metric variables. Once these are established, it is a routine exercise to derive the fall-off rate for the conformal variable $\Omega \sim 1 + \mathcal{O}(1/r)$ [5].

As a last remark on this subject, we note that usual proofs of Birkhoff's theorem do not utilize explicitly the equation of motion $\dot{\pi}^{ab}$. This equation determined a leftover integration constant $b = 1$ in the lapse (4) from (15) and allowed us to recover exactly the isotropic black hole space-time.

Properties of the solution

The first thing to notice is that this solution is valid under our assumptions wherever there is no matter. We find no reason to disregard this solution as non-physical for Shape Dynamics. Although the lapse is gauge-dependent in both GR and in Shape Dynamics, the breaking of the *duality* between the two theories *can* be signaled by the collapse of the lapse. The unique lapse solution N_o given in the text is required to make contact with a space-time, since for the duality we must set $\Omega = 1$, and this lapse definitely breaks down at the horizon. If one would try to interpret this solution as a *space-time*, one obtains that the volume degenerates at the horizon, the 4-metric becomes degenerate, and thus does not possess an inverse.⁶ These statements are invariant under space-time diffeomorphisms, and thus this solution should not be considered as a space-time per se, but instead what one would obtain if attempting to describe a valid Shape Dynamics solution in space-time language.

In other words, this is *not* a solution for GR. It can be a solution only *separately* for the intervals $r > m/2$ and $r < m/2$. While the isotropic Schwarzschild solution is considered a solution for GR only for $r > m/2$, for Shape Dynamics it should be considered as physical, since no observables of Shape Dynamics (spatial conformal diffeomorphism invariant quantities) break down at the horizon. Furthermore, at the no back-reaction level, we can use the “reconstructed space-time” as a background for the motion of particles in Shape Dynamics. This entails that we can calculate anything that we would normally calculate in GR, keeping in mind that Lorentz-invariance - which is not a *fundamental* symmetry of Shape Dynamics - can be broken at the horizon.

Let us then mention a few of the features of this solution seen as a space-time, reproduced by us, but originally made in [8]:

- An observer in the asymptotically flat region will not see any difference between this solution and a Schwarzschild black hole. A simple calculation shows that the infalling radial geodesic takes infinite proper time to reach $r = 0$. This is of course a physical, observable distinction between a Schwarzschild solution and this solution.
- The infalling observer will (after finite proper time) experience falling into a mirror universe. The mirror universe is obtained by replacing the isotropic radial coordinate with $r \rightarrow m^2/(4r)$, which can be checked

⁶One should not confuse this *space-time* degeneracy of the metric with the usual degeneracy of the horizon, which is topologically a 3-surface but metrically a 2-surface. The space-time metric is, in every description of horizons, still *non degenerate* at the horizon.

to leave the form of the line element (16) invariant. Thus the solution has the property of *inversion*, associated to conformal invariance (as for instance in the method of images in electrodynamics).

- In this manner the boundaries of the left and right parts of the usual Kruskal diagram are identified, and the singularity does not figure anywhere. Notice that the mirror universe essentially decompactifies the region near $r = 0$ by a conformal transformation that puts infinite spatial distance between $r = 0$ and any other point in a given $t = \text{const.}$ surface.
- One might worry about Penrose’s singularity theorem. However, there is a discontinuity of the expansion scalar at the event horizon of the wormhole, which prevents the expansion scalar from decreasing to $-\infty$, as it would occur inside a Schwarzschild black hole. This discontinuity guarantees that timelike geodesics in the gravitational field of a wormhole are complete.⁷

Clearly, the physical implications of this model depend crucially on whether we can simulate real collapse of a given distribution of matter. This and other aspects, we leave to upcoming work.

Nonetheless, it is interesting to (wildly) speculate on some possible consequences for this result. For example, suppose that isolated particles could be modeled by such defects.⁸ This would mean that after traversing the particle’s Schwarzschild radius there is no sense in which decreasing the radius would get us closer to the particle. The “closer” we get to the particle the less we would feel its gravitational influence.

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⁷We should mention that space-times in maximal slicing - which have a good translation into solutions of Shape Dynamics - have a well-known singularity avoidance property, for its Eulerian observers, which are the natural observers to take in the Shape Dynamics side.

⁸This is of course a terrible idea for any physical elementary particle. The problem is that the ratio between the electric charge and the mass of an electron e/m don’t obey the bounds of Reissner-Nordstrom solutions (and thus would contain a naked singularity). An analogous objection holds if one considers spins and the ratio between the angular momentum and the mass J/m .